Georg Fantner

Week 3
Strain and stress in one dimension

1. Treatment of bars like springs
2. Stress in axially loaded bars
3. Microscopic equilibrium
4. Statically indeterminate systems - The displacement (stiffness) method

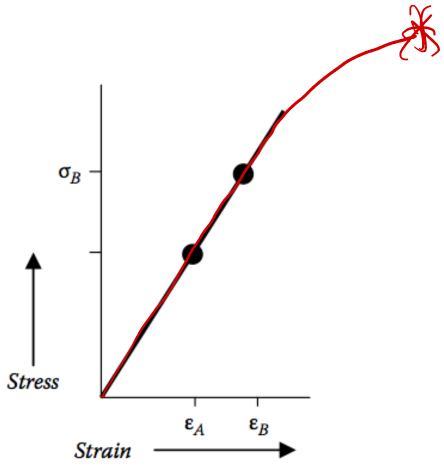
Speaker

Stress-Strain relationships: Hooke's law

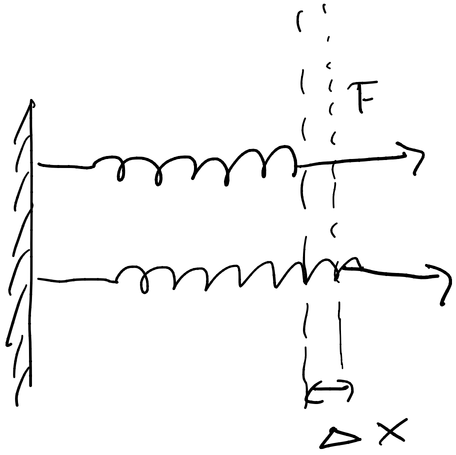
What is the relationship between the applied load and the deformation of a structure?

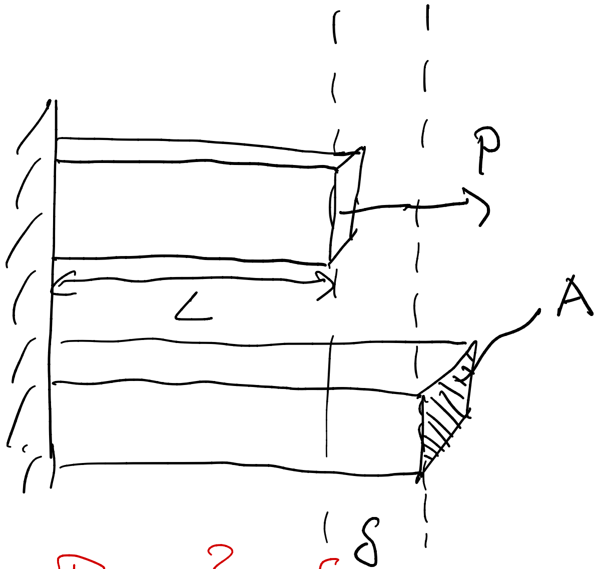
Stress Strain relationships

- For each incremental stress there is a proportional increase of strain $\sigma \propto \varepsilon$
- Hooke's law for normal stress: $$\sigma = E \cdot \varepsilon$$
- *E = "Youngs modulus"* or *"elastic modulus"*
- Hooke's law for shear stress: $$\tau = G \cdot \gamma$$
- *G="shear modulus"* or *"modulus of rigidity"*



- Proportionality limit: the limit of where the increase in stress becomes non-linear with the increase in strain





$$F = k \, \Delta x$$

$$P = \, ? \, \delta$$

$$\sigma := \frac{P}{A} \qquad \varepsilon = \frac{\delta}{L}$$

HOOKES LAW: $\sigma = E \cdot \varepsilon$

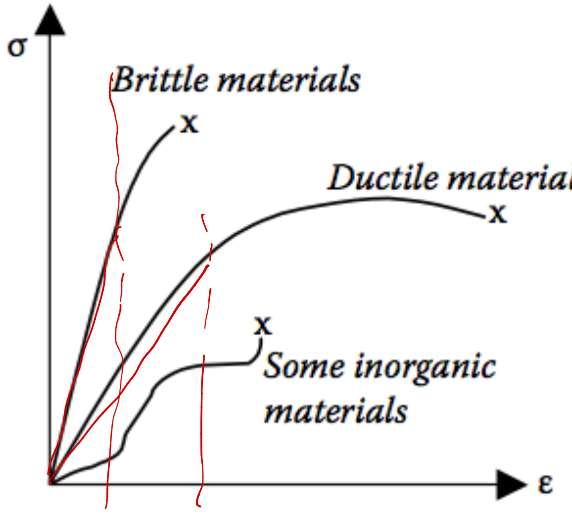
↳ Young's modulus, Elastic modulus.

Assumptions we are making in the simple form of Hooke's law

- The material is homogeneous: E and G do not vary from point to point
- The material isotropic: E and G are invariant with respect to any rotation of the coordinate system
- There is no effect of temperature on the mechanical properties E and G

Materials classification:

- Ductile materials: materials that can withstand a large amount of strain without significant increase in stress (rubber, polymers, skin, low carbon steel)
- Brittle materials: materials that experience a huge increase in stress for even small strains. These materials will fail abruptly after small amounts of deformation (ceramics, silicon, cast iron, concrete)
- Some materials have different properties in different directions (wood, bone, carbon composites). Such materials CAN NOT be treated with the simple form of Hooke's law.



Deformation in axially loaded bars

- Hooke's Law:

$$\sigma = E \cdot \varepsilon$$

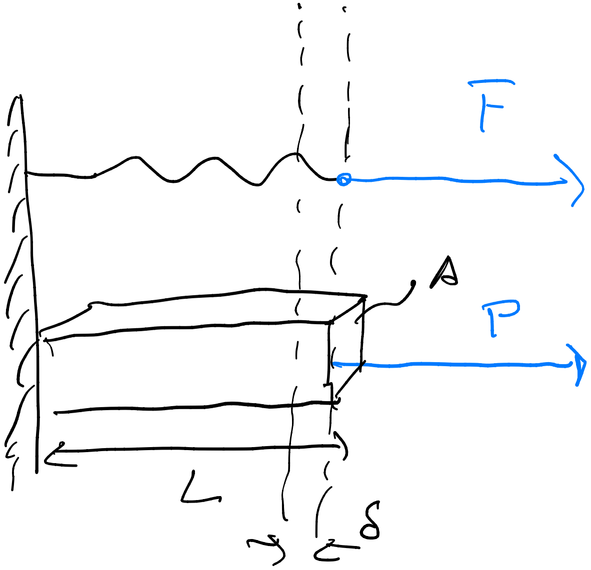
$$\frac{P}{A} = E \cdot \frac{\delta}{L}$$

- Strain in axially loaded bar with uniform cross-section:

$$\delta = \frac{PL}{AE} \qquad P = \underbrace{\frac{AE}{L}}_{\text{stiffness of bar}} \cdot \delta$$

- Strain in axially loaded bar with arbitrary material or cross-section:

$$\delta = \int_0^L \frac{P(x)}{A(x)E(x)} dx$$



$$F = k \cdot \delta$$

$$\sigma = E\varepsilon \qquad \sigma := \frac{P}{A} \qquad \varepsilon = \frac{\Delta L}{L} = \frac{\delta}{L}$$

$$\frac{P}{A} = E \cdot \delta / L$$

$$\boxed{P = \underbrace{\frac{EA}{L}} \cdot \delta}$$

MACROSCOPIC EQN. FOR BAR IN TENSION

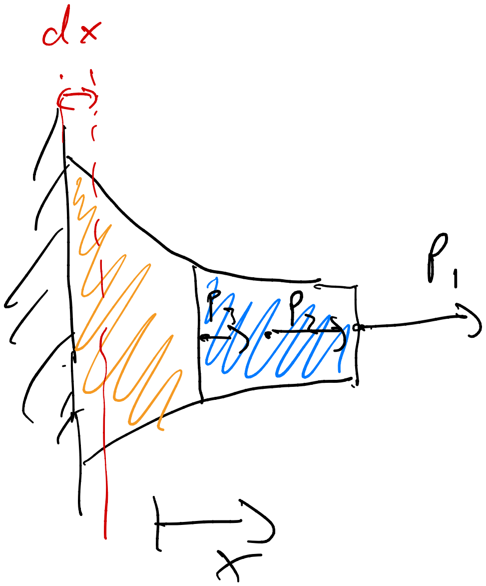
$k :=$ STIFFNES OF BAR

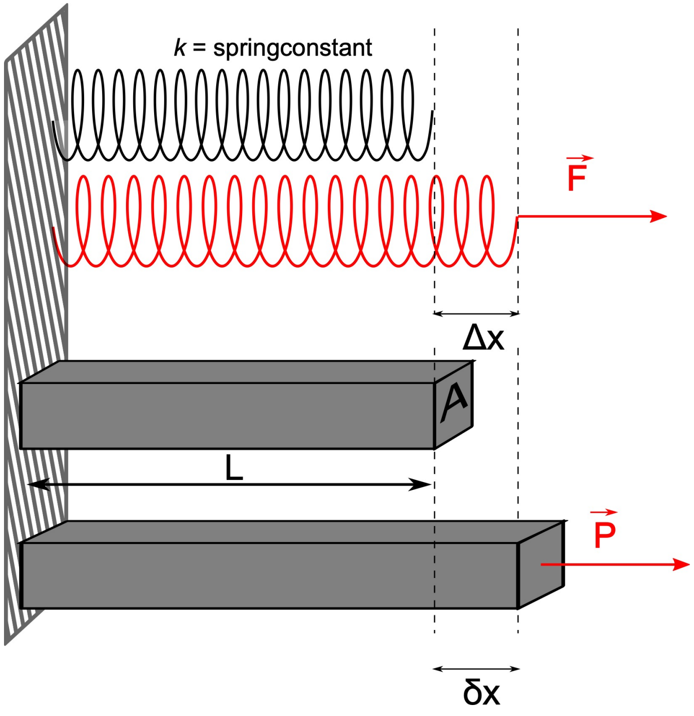
FLEXIBILITY: $f = \frac{1}{k} = \frac{L}{AE}$

$$P = \frac{EA}{L} \cdot \delta$$

$$\Rightarrow \delta = \frac{L}{EA} \cdot P$$

$$\boxed{\delta = \int_0^L \frac{P(x)}{E(x)\,A(x)}\, dx}$$





Treatment of bodies like springs

From the axially loaded bar we have seen the communality of the load extension curve to the basic force extension curve of a spring:

	Spring	Axially loaded bar
Hooke's Law	$F = k \cdot \Delta x$	$P = \frac{AE}{L} \cdot \delta$
Spring constant	k	$k = \frac{AE}{L}$

Stress in axially loaded bars

What happens inside the bar when it is stretched?

- Axially loaded bar fixed at x=0 and loaded by force P at x=L (figure a)
- Apply method of section normal to bar axis (figure b)

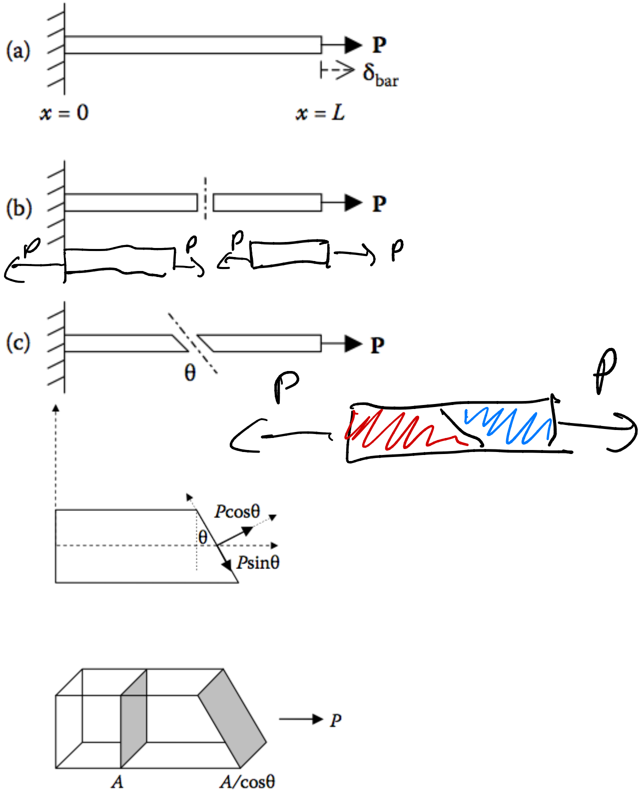
$$\sigma \equiv \frac{P}{A}$$

- Apply method of section angled to bar axis (figure c)

Angular dependence of stress

$$\sigma_\theta = \frac{force}{area} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta$$

$$\tau_\theta = -\frac{P\sin\theta}{A/\cos\theta} = -\frac{P}{A}\sin\theta\cos\theta$$



Stress in axially loaded bars

Directions of maximum stress

- Maximum normal stress:

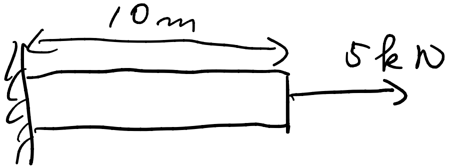
$$\sigma_\theta = \frac{P}{A}\cos^2\theta \rightarrow \boxed{\sigma_{max} = \frac{P}{A}} @\,\theta = 0$$

- Maximum shear stress:

$$\frac{d}{d\theta}\tau_\theta = \frac{d}{d\theta}\left(-\frac{P}{A}\sin\theta\cos\theta\right) = 0 \rightarrow \theta = 45^o$$

$$|\tau_{max}| = \frac{P}{A}\sin(\pi/4)\cos(\pi/4)$$

$$\boxed{|\tau_{max}| = \frac{P}{2A} = \frac{\sigma_{max}}{2}}$$



Given: • Length: $L = 10m$

• Load : $P = 5kN$

• Geometry: round $\Rightarrow A = \pi r^2$

• Max strain ε_{max}: 3 mm

• Max stress σ_{max}: $150 MN/m^2$

• $E = 200\ GPa$

Asked:

ϕ_{min}

Gov. Eqn:

- Hookes Law
- Bar in tension formula

$$P = k\delta = \frac{AE}{L}$$

Example: elongation of a steel bar

A load of 5kN pulls on a round steel bar (E=200GPa) that has a length of 10m. The bar is not allowed to stretch more than 3mm, nor must it exceed a stress of 150MN/m². What is the minimum diameter that the bar must have?

ANSWER :

σ_{MAX}:

$$\sigma_{MAX} = \frac{P}{A_{min}} \Rightarrow A_{min} = \frac{P}{\sigma_{MAX}} = \frac{5 \cdot 10^3}{150 \cdot 10^6} = 3.33 \cdot 10^{-5}\, m^2$$

$$r_{min} = \sqrt{\frac{A}{\pi}} = 3.3\ mm$$

δ_{MAX}:

$$P = \frac{AE}{L}\,\delta_{MAX}$$

$$A = \frac{L}{\delta_{MAX} \cdot E}\,P = \frac{10\,m}{3 \cdot 10^{-3}\,m \cdot 200 \cdot 10^9\,Pa} \cdot 5 \cdot 10^3\,Pa = 8.33 \cdot 10^{-5}\,m^2$$

$$r_{min} = 5.15\ mm$$

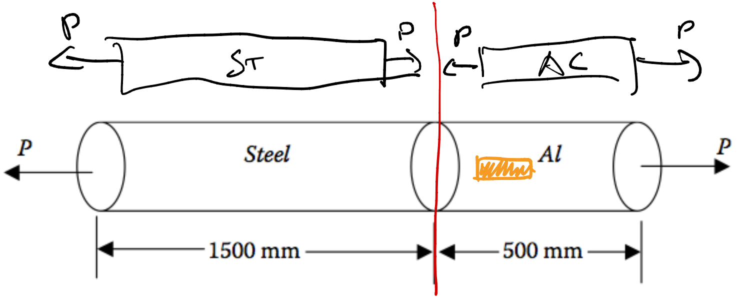
LIMIT IS DUE TO MAX EXTENSION: $d_{min} = \underline{10.3\ mm}$

Example: Multi-material composite bar

A solid bar 50 mm in diameter and 2000 mm long consists of a steel and an aluminum section, as shown below. When an axial force *P* is applied to the system, a strain gauge attached to the aluminum indicates an axial strain of 0.000873 m/m. The elastic modulus of steel is $=E_{ST}=200GPa$, $E_{ALU}=70GPa$.

(a) Determine the magnitude of applied force *P,* and

(b) if the system behaves elastically, find the total elongation of the bar.



given: - GEOMETRY

- $L_{ST} = 1.5\,m$
- $L_{AL} = 0.5\,mm$

- MATERIAL

- $E_{AL} = 70\,GPa$
- $E_{ST} = 200\,GPa$

- STRAIN IN AL: $\varepsilon_{AL} = 8.73 \cdot 10^{-4}$

ASKED:

a) P

b) ΔL TOTAL

EQN: Bar in Tension:

$$\delta = \int_0^L \frac{P(x)}{E(x)\cdot A(x)}\,dx$$

ANSWER

$$\sigma = \frac{P}{A} \Rightarrow P = \sigma \cdot A = E \cdot \varepsilon \cdot A$$

$$= E \cdot \varepsilon \cdot \pi \left(\frac{d}{2}\right)^2$$

$$= 70 \cdot 10^9 \cdot 8.73 \cdot 10^{-4} \cdot \pi \cdot (0.025\,m)^2$$

$$= 120\,kN$$

b: $$\delta_{TOT} = \delta_{ST} + \delta_{AL} = \int_0^{L_{ST}} \frac{P}{E_{ST} A} dx + \int_{L_{ST}}^{L} \frac{P}{E_{AL} A} dx$$

$$= \frac{Px}{E_{ST} A}\Big|_0^{1.5} + \frac{Px}{E_{AL} \cdot A}\Big|_{1.5}^{2}$$

$$= \frac{P}{A}\left(\frac{1.5}{E_{ST}} + \frac{0.5}{E_{AL}}\right) = \frac{120\,kN}{1.96\cdot 10^{-3}\,m^2}\left(\frac{1.5}{200\cdot 10^9} + \frac{0.5}{70\cdot 10^9}\right)$$

$$\delta_{TOT} = 0.897\,mm \Rightarrow \boxed{\delta_{TOT} = 0.90\,mm}$$

Microscopic equilibrium

We have defined strain in microscopic terms, we have defined Hookes law in terms of stress and strain, now we need to define equilibrium in microscopic terms.

BULK MICROSCOPIC

"DIMENSION" $\Delta L \Leftrightarrow \varepsilon = \frac{du}{dx}$

KINEMATIC EQN

LOAD: $F = k \cdot \Delta x \Leftrightarrow \sigma = E \cdot \varepsilon$

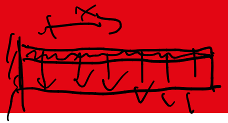
CONSTITUTIVE EQN

EQUILIBR: $\sum F = 0 \Leftrightarrow$?

EQUILIBRIUM EQN.

What kind of forces can act on an object?

Point	Point loads P: A discrete force acting at one point and one point only
Distributed	Distributed axial loads q(x): force that acts per unit length
Body	Body forces B(x): force that acts per volume (e.g. gravity, magnetic field etc.)

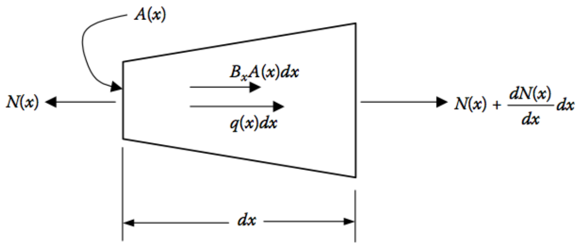


Georg Fantner

Microscopic equilibrium analysis

Calculation of the stress at an arbitrary point in the structure acting on an infinitesimal element dV=A(x)dx

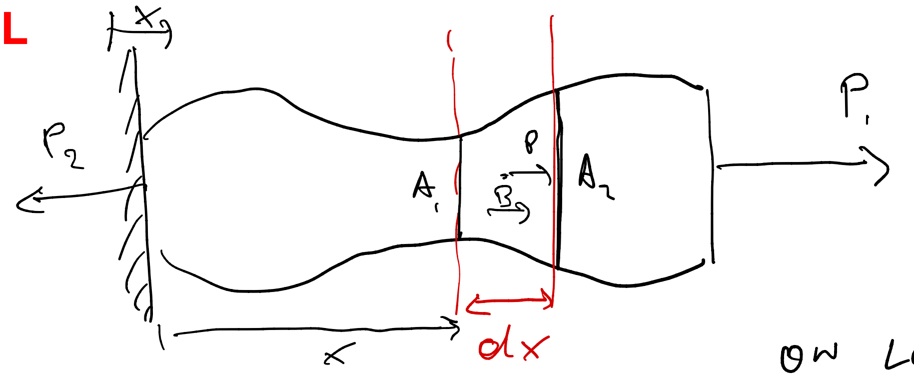
The *internal* axial force N(x) balances both *external* forces (distributed axial load q(x), and axial body force $B_{x)}$



$$\left\{N(x) + \frac{dN(x)}{dx} \cdot dx\right\} - N(x) + q(x)dx + B_x A(x)dx = 0$$

$$\frac{dN(x)}{dx} + q(x) + B_x A(x) = 0$$

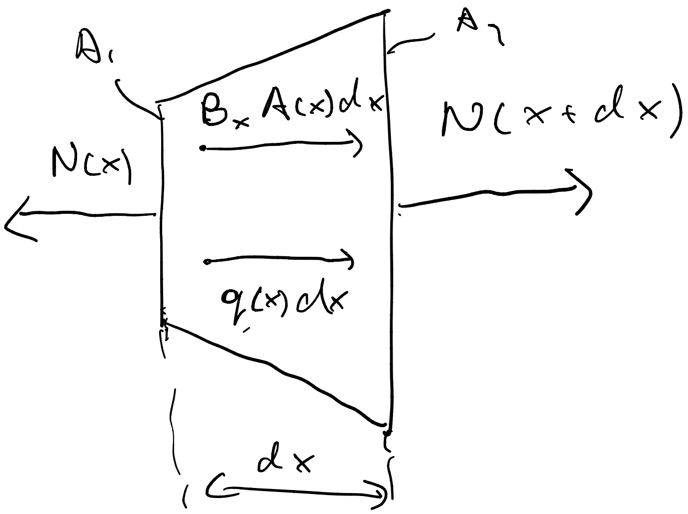
First order, ordinary differential equation for the axial normal stress



$$\sum F = +P_1 - P_2 = 0 \qquad P_1 = P_2$$

ON LEFT: $N(x)$

RIGHT: $N(x+dx) \approx N(x) + \frac{dN(x)}{dx} \cdot dx$



EXTERNAL FORCES:

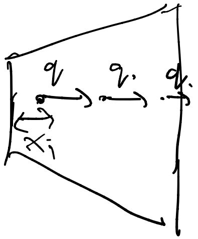
BODY FORCE: $B_x \cdot V_{(x)} = B_x \cdot A(x) \cdot dx$

DISTR. FORCE: $q(x) \cdot dx$

EQUILIBRIUM:

$$\sum F_x = -\cancel{N(x)} + B_x A(x)dx + q(x)dx + \cancel{N(x)} + \frac{dN(x)}{dx}\cdot dx = 0$$

$$\frac{dN(x)}{dx} + \underbrace{q(x)} + B_x A(x) =$$



$$\frac{dN(x)}{dx} + \sum_i P_i\, \delta(x-x_i) + B_x A(x) = 0$$

The three equations

In structural mechanics, we (always) rely on these 3 equations:

- Equilibrium equation:
  ensures that all forces are in equilibrium

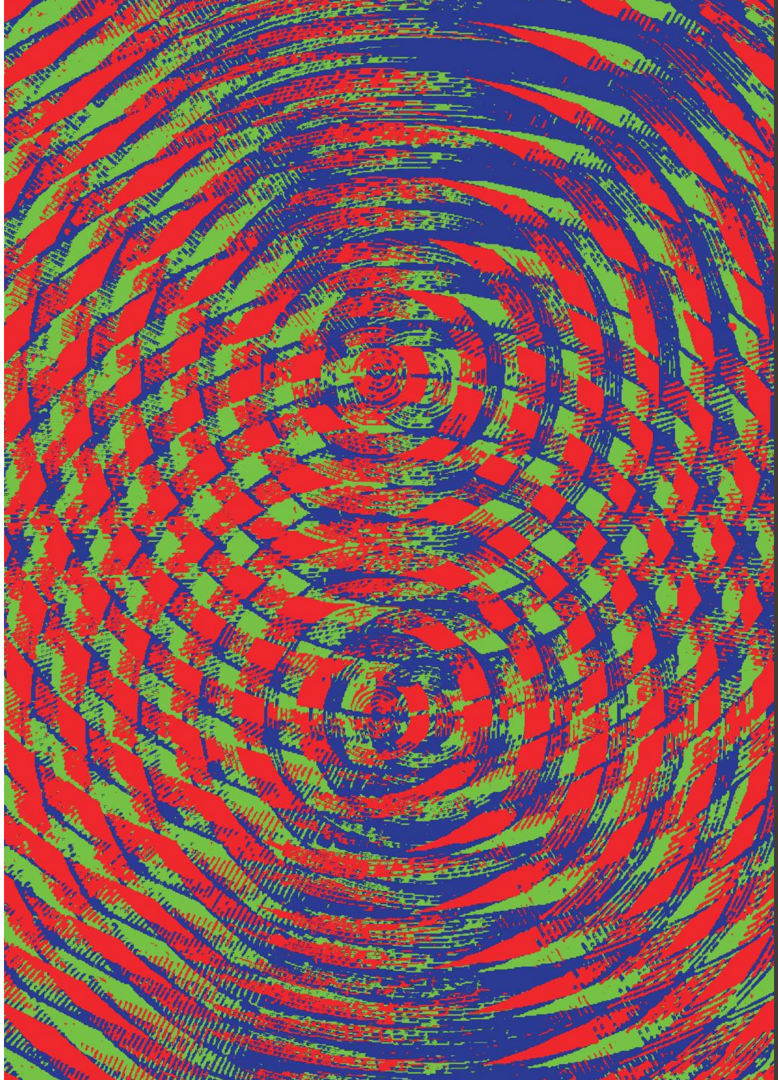
$$\frac{dN(x)}{dx} + \sum_i P_i \delta(x - x_i) + B_x A(x) = 0$$

- Constitutive equation:
  Relates two quantities with materials specific properties:

$$E = \frac{\sigma}{\varepsilon}$$

- Kinematic Equation:
  relates strain (ε) to displacement (u):

$$\varepsilon(x) = \frac{du(x)}{dx}$$



Superposition principle

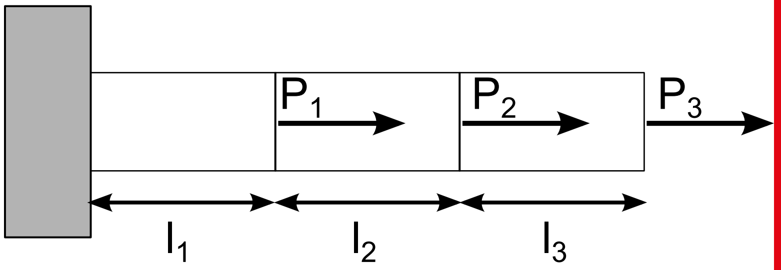
The effect of the sum of forces is equal to the sum of the effects of the individual forces.

$$F(x_a + x_b) = F(x_a) + F(x_b)$$

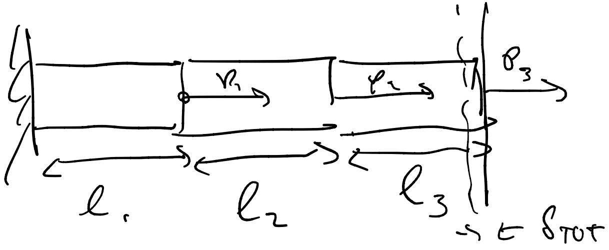
$$F(a * x_a) = a * F(x_a)$$

This principle is only valid whenever there exists a linear relationship between the external loads and the structural displacement! In our case we deal only with small displacements where Hooke's law is valid. So the superposition principle is valid.

Example: superposition



A uniform bar with cross-section A and elastic modulus E is loaded at 3 points along its axis as shown below. What is the total elongation of the bar?



$l_1 = l_2 = l_3$

- P_1: $\delta_{\text{due to } P_1} = \dfrac{P_1 l_1}{AE}$

- P_2 $\delta_{\text{DUE TO } P_2} = \dfrac{P_2 (l_1 + l_2)}{AE} = \dfrac{2 P_2 l}{AE}$

- P_3 $\delta_{\text{DUE TO } P_3} = \dfrac{P_3 (l_1 + l_2 + l_3)}{AE} = \dfrac{3 P_3 l}{AE}$

IF $P_1 = P_2 = P_3$

$\delta_{TOT} = \delta_{P_1} + \delta_{P_2} + \delta_{P_3}$

$$\boxed{\delta_{TOT} = 6 \frac{Pl}{AE}}$$